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On a Problem that Does not Have Basis Property of Root Vectors, Associated with a Perturbed Regular Operator of Multiple Differentiation

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Abstract. A spectral problem for a multiple differentiation operator with integral perturbation of boundary value conditions which are regular but not strongly regular is considered in the paper. The feature of the problem is the absence of the basis property of the system of root vectors. A characteristic determinant of the spectral problem is constructed. It is shown that absence of the basis property of the system of root functions of the problem is unstable with respect to the integral perturbation of the boundary value condition.

Keywords: multiple differentiation operator, integral perturbation of boundary value conditions, basis property, root vectors, system of eigenfunctions and associated functions, eigenvalue, characteristic determinant.

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1. Introduction and formulation of the problem

It is well known that a system of eigenfunctions of an operator defined by a formally self-adjoint differential expression with arbitrary self-adjoint boundary value conditions, that provide a discrete spectrum, forms an orthonormal basis in the space L_2. The problem of preservation of basis properties with respect to some weak (in a certain sense) perturbation of the initial operator was investigated in many papers. For example, the problem was studied in the case of a self-adjoint initial operator [1–3], and in the case of a non self-adjoint initial operator [4–6]. The following spectral problem is considered in this paper (it is close to the problems considered in [1, 4, 7])

$$l(u) \equiv -u''(x) = \lambda u(x), \quad 0 < x < 1 \tag{1}$$

$$U_1(u) \equiv u'(0) - u'(1) - \alpha u(1) = 0, \quad \alpha > 0, \tag{2}$$

$$U_2(u) \equiv u(0) = \int_0^1 \overline{p(x)} u(x) dx, \quad p(x) \in L_2(0,1). \tag{3}$$

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Where λ is an arbitrary complex number. Stability of the basis property of root vectors of the spectral problem has was studied in the case $\alpha = 0$ [8]. It is common knowledge that system of root functions of an ordinary differential operator with arbitrary strongly regular boundary value conditions forms Riesz basis in the space $L_2(0,1)$. In the case when boundary value conditions are regular but not strongly regular, the basis property of systems of root functions, in contrast to the completeness property, is not even determined by the specific type of boundary value conditions. This effect was first established by V. A. Il'in [5], and the corresponding example was constructed for some second-order differential operator. In this case it was shown that not only boundary value conditions but also the values of coefficients of the differential operator affect the existence of the basis property. Moreover, this property can be changed with an arbitrarily small change in values of the coefficients in the metric of those classes in which these coefficients are given. Let L_1 be an operator in $L_2(0,1)$. It is defined by (1) and "perturbed" boundary value conditions

$$U_1(u) \equiv u'(0) - u'(1) - \alpha u(1) = 0, \quad \alpha > 0, \quad U_2(u) \equiv u(0) = \int_0^1 \overline{p(x)} u(x) dx, \quad p(x) \in L_2(0,1).$$

Unperturbed operator (the case $p(x) = 0$) is denoted by L_0. Various variants of the integral perturbation of boundary value conditions were considered [9–13]. It was assumed that unperturbed operator L_0 has a system of eigenfunctions and associated functions (E&AF) that forms the Riesz basis in $L_2(0,1)$. Characteristic determinant of the spectral problem for operator L_1 was also constructed. Then inferences were drawn about stability or instability of the Riesz basis property of the E&AF problem with an integral perturbation of the boundary value condition.

The distinctive feature of this paper is that system of eigenfunctions of unperturbed problem (1)– (3) is complete but it does not form a basis in $L_2(0,1)$ [14]. Therefore, the method that we use in the previous papers cannot be applied in this case.

2. Characteristic determinant of the spectral problem

Boundary value conditions in problem (1)–(3) are regular but not strongly regular [15]. System of root functions of operator l is a complete system but does not even form a usual basis in $L_2(0,1)$ [14]. However, as it was shown in [16], a basis can be constructed with the use of these eigenfunctions. This makes it possible to apply the method of separation of variables to solve the initial-boundary value problem with boundary value condition (2). The problem that does not have the basis property of root vectors in $L_2(0,1)$ defined by expression (1) with integral perturbation of boundary value condition (2) is studied [17]. The general solution of equation (1) has the form

$$u(x,\lambda) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x.$$

Using boundary value conditions (2)–(3), we obtain the following linear system with respect to coefficients C_k :

$$\begin{cases} C_1\left(\sqrt{\lambda}\sin\sqrt{\lambda} - \alpha\cos\sqrt{\lambda}\right) + \left(\sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \alpha\sin\sqrt{\lambda}\right) = 0, \\ C_1\left(1 - \int_0^1 \overline{p(x)}\cos\sqrt{\lambda}x dx\right) + C_2\int_0^1 \overline{p(x)}\sin\sqrt{\lambda}x dx = 0. \end{cases} \tag{4}$$

Therefore, characteristic determinant of problem (1)–(3) has the form

$$\Delta_1(\lambda) = \left(\sqrt{\lambda}\sin\sqrt{\lambda} - \alpha\cos\sqrt{\lambda}\right)\cdot\int_0^1 \overline{p\,(x)}\sin\sqrt{\lambda}x dx - \\ - \left(\sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right) - \alpha\sin\sqrt{\lambda}\right)\left(1 - \int_0^1 \overline{p\,(x)}\cos\sqrt{\lambda}x dx\right). \quad (5)$$

If $p(x) = 0$ then the characteristic determinant of unperturbed problem (1)–(3) is obtained. It is denoted by

$$\Delta_0(\lambda) = \sqrt{\lambda}\left(1-\cos\sqrt{\lambda}\right) - \alpha\sin\sqrt{\lambda}.$$

Solving the equation $\Delta_0(\lambda) = 0$, we have two series of eigenvalues $\lambda_k^{(1)} = (2\pi k)^2$, $k = 1, 2, \ldots$, $\lambda_k^{(2)} = (2\beta_k)^2$, $k = 1, 2, \ldots$, of unperturbed problem (1)–(3). Here β_k are roots of the equation

$$\operatorname{tg}\beta = \frac{\alpha}{2\beta}, \quad \beta > 0. \quad (6)$$

They are positive and satisfy the following inequalities:
$\pi k < \beta_k < \pi k + \dfrac{\pi}{2}$, $k = 0, 1, 2, \ldots$.
For the difference $\delta_k = \beta_k - \pi k$ at large enough k the following two-sided estimates are true

$$\frac{\alpha}{2\pi k}\left(1 - \frac{1}{2\pi k}\right) < \delta_k < \frac{\alpha}{2\pi k}\left(1 + \frac{1}{2\pi k}\right). \quad (7)$$

When $p(x) = 0$ eigenfunctions of the unperturbed problem have the following form

$$u_k^{(1)}(x) = \sin(2\pi k x), \quad k = 1, 2, \ldots, \quad u_k^{(2)}(x) = \sin(2\beta_k x), \quad k = 1, 2, \ldots .$$

When $p(x) = 0$ let us consider the problem adjoint to unperturbed problem (1)–(3)

$$l*(\nu) = \overline{\lambda}\nu, v'(1) + \alpha v(0) = 0; \quad v(0) - v(1) = 0 \quad (8)$$

It has eigenfunctions

$$v_k^{(1)}(x) = C_k^{(1)}\left(\cos(2\pi k x) - \frac{\alpha}{2\pi k}\sin(2\pi k x)\right), \quad k = 1, 2, \ldots, \\ v_k^{(2)}(x) = C_k^{(2)}\left(\cos(2\beta_k x) + \frac{\alpha}{2\beta_k}\sin(2\beta_k x)\right), \quad k = 1, 2, \ldots,$$

where $C_k^{(1)}, C_k^{(2)}$ are selected from the ratio of biorthogonality $\left(u_k^{(1)}, v_k^{(1)}\right) = 1, \left(u_k^{(2)}, v_k^{(2)}\right) = 1$. Hence,

$$C_k^{(1)} = -\frac{4\pi}{\alpha}, \quad C_k^{(2)} = \frac{4\pi}{\alpha} + O\left(\frac{1}{k}\right). \quad (9)$$

An auxiliary system was constructed that forms the basis in $L_2(0,1)$ [16]

$$u_0(x) = u_0^{(2)}(x)\cdot(2\beta_0)^{-1}, \quad u_{2k}(x) = u_k^{(1)}(x), \\ u_{2k-1}(x) = \left(u_k^{(2)}(x) - u_k^{(1)}(x)\right)\cdot(2\delta_k)^{-1}, \quad k = 1, 2, \ldots .$$

The system that is biorthogonal to auxiliary system is

$$v_0(x) = 2\beta_0 v_0^{(2)}(x), \quad v_{2k}(x) = v_k^{(2)}(x) + v_k^{(1)}(x), \quad v_{2k-1}(x) = 2\delta_k v_k^{(2)}(x), \quad k = 1, 2, \ldots .$$

It is constructed from the eigenfunctions of problem (8). The function $p(x)$ can be represented in terms of Fourier series based on the auxiliary system $\{v_k(x)\}$:

$$p(x) = a_0 v_0(x) + \sum_{k=1}^{\infty} (a_k v_{2k}(x) + b_k v_{2k-1}(x)). \tag{10}$$

We integrate expression (5) and obtain

$$\int_0^1 \overline{p\,(x)} \sin\sqrt{\lambda} x dx = \Delta_0(\lambda) \cdot \left\{ \bar{a}_0 \cdot C_0^{(2)} \frac{2\beta_0}{\lambda - (2\beta_0)^2} + \right.$$
$$\left. + \sum_{k=1}^{\infty} \bar{a}_k \cdot C_k^{(1)} \frac{1}{\lambda - (2k\pi)^2} + \sum_{k=1}^{\infty} \frac{C_k^{(2)}}{\lambda - (2\beta_k)^2} \left(\bar{a}_k + \bar{b}_k(2\delta_k)\right) \right\},$$

where

$$\Delta_0(\lambda) = \sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \alpha \sin\sqrt{\lambda};$$

$$\int_0^1 \overline{p\,(x)} \cos\sqrt{\lambda} x dx = \bar{a}_0 \frac{(2\beta_0) \cdot C_0^{(2)}}{\lambda - (2\beta_0)^2} \left(\sqrt{\lambda}\sin\sqrt{\lambda} - \alpha\cos\sqrt{\lambda}\right) +$$
$$+ \sum_{k=1}^{\infty} \bar{a}_k \frac{C_k^{(1)}}{\lambda - (2k\pi)^2} \left(\alpha\left(1 - \cos\sqrt{\lambda}\right) + \sqrt{\lambda}\sin\sqrt{\lambda}\right) +$$
$$+ \sum_{k=1}^{\infty} \left(\bar{a}_k + \bar{b}_k(2\delta_k)\right) \frac{C_k^{(2)}}{\lambda - (2\beta_k)^2} \left(\sqrt{\lambda}\sin\sqrt{\lambda} - \alpha\cos\sqrt{\lambda} - \alpha\right).$$

Using obtained results, determinant (5) is reduced to the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \left[-1 + \alpha \sum_{k=1}^{\infty} \left(a_k \left(C_k^{(1)} \frac{1}{\lambda - (2k\pi)^2} - C_k^{(2)} \frac{1}{\lambda - (2\beta_k)^2} \right) - \right.\right.$$
$$- b_k \frac{2\delta_k C_k^{(2)}}{\lambda - (2\beta_k)^2} \Big) + 2\left(\sqrt{\lambda}\sin\sqrt{\lambda} - \alpha\cos\sqrt{\lambda}\right) \cdot \left(a_0 \frac{(2\beta_0) \cdot C_0^{(2)}}{\lambda - (2\beta_0)^2} + \right.$$
$$\left.\left. + \sum_{k=1}^{\infty} \left(a_k \left(C_k^{(1)} \frac{1}{\lambda - (2k\pi)^2} + C_k^{(2)} \frac{1}{\lambda - (2\beta_k)^2} \right) + b_k \frac{(2\delta_k) C_k^{(2)}}{\lambda - (2\beta_k)^2} \right) \right) \right], \tag{11}$$

where $\Delta_0(\lambda) = \sqrt{\lambda}\left(1 - \cos\sqrt{\lambda}\right) - \alpha\sin\sqrt{\lambda}$.

Expression in square brackets is denoted by $A(\lambda)$. The obtained above result is formulated as the following theorem.

Theorem 2.1 *Characteristic determinant of spectral problem* (1)–(3) *with perturbed boundary value conditions can be represented in form* (11), *where* $\Delta_0(\lambda)$ *is the characteristic determinant of the unperturbed problem, and* a_k, b_k *are the Fourier coefficients of expansion* (10) *of the function* $p(x)$ *in the biorthogonal system of eigenfunctions of adjoint problem* (8).

Function $A(\lambda)$ with $\lambda = (2\beta_k)^2$ and $\lambda = (2\pi k)^2$ has first order poles. Therefore, $\Delta_1(\lambda) = \Delta_0(\lambda) \cdot A(\lambda)$ is an entire analytic function of the variable λ. Characteristic determinant (11)

has the simple form when $p(x)$ is represented as a finite sum in (10). When there exists a number N such that $a_k = b_k = 0$ for all $k > N$ then function $A(\lambda)$ has the form

$$A(\lambda) = -1 + \alpha \sum_{k=1}^{N} \left[a_k \left(C_k^{(1)} \frac{1}{\lambda - (2k\pi)^2} - C_k^{(2)} \frac{1}{\lambda - (2\beta_k)^2} \right) - \right.$$
$$\left. - \, b_k \frac{2\delta_k C_k^{(2)}}{\lambda - (2\beta_k)^2} \right] + 2 \left(\sqrt{\lambda} \sin \sqrt{\lambda} - \alpha \cos \sqrt{\lambda} \right) \cdot \left(a_0 \frac{(2\beta_0) \cdot C_0^{(2)}}{\lambda - (2\beta_0)^2} \, + \right.$$
$$\left. + \sum_{k=1}^{N} \left[a_k \left(C_k^{(1)} \frac{1}{\lambda - (2k\pi)^2} + C_k^{(2)} \frac{1}{\lambda - (2\beta_k)^2} \right) + b_k \frac{(2\delta_k) C_k^{(2)}}{\lambda - (2\beta_k)^2} \right] . \quad (12)$$

It follows from (12) that $\Delta_1(\lambda_k^1) = \Delta_1(\lambda_k^2) = 0$, for all $k > N$. Consequently, all eigenvalues $\lambda_k^1, \ \lambda_k^2, \ k > N$ of unperturbed problem (1)–(3) ($p(x) = 0$) are eigenvalues of perturbed problem (1)–(3). Multiplicity of eigenvalues $\lambda_k^1, \ \lambda_k^2, \ k > N$ is preserved. It follows from the condition of biorthogonality of the system of eigenfunctions $\left\{ u_k^{(1)}(x), \ u_k^{(2)}(x) \right\}$ and $\left\{ v_k^{(1)}(x), \ v_k^{(2)}(x) \right\}$ that $\int\limits_0^1 \overline{p\,(x)} u_k^{(1)}(x) dx = 0, \ \int\limits_0^1 \overline{p\,(x)} u_k^{(2)}(x) dx = 0, \ k > N$. Thus, eigenfunctions $\left\{ u_k^{(1)}(x), \ u_k^{(2)}(x) \right\}, \ k > N$ of unperturbed problem ($p(x) = 0$) (1)–(3) satisfy the boundary value conditions of perturbed problem (1)–(3). Therefore, the system of eigenfunctions of perturbed problem (1)–(3) and the system of eigenfunctions of the unperturbed problem ($p(x) = 0$) coincide which do not form a basis with the exception of a finite number of the first terms. Then the system of eigenfunctions of perturbed problem (1)–(3) is also not a basis in $L_2(0, 1)$ in this particular case.

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Об одной задаче, не обладающей свойством базисности корневых векторов, связанной с возмущенным регулярным оператором кратного дифференцирования

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Аннотация. В работе рассматривается спектральная задача для оператора кратного дифференцирования при интегральном возмущении краевых условий одного типа, являющихся регулярными, но не усиленно регулярными, где особенностью задачи считают отсутствие свойства базисности у системы корневых векторов. Построен характеристический определитель спектральной задачи. Показано, что отсутствие базисности у системы корневых функций задачи неустойчиво относительно интегрального возмущения краевого условия.

Ключевые слова: оператор кратного дифференцирования, интегральное возмущение краевых условий, базисность, корневые векторы, система собственных и присоединенных функций, собственное значение, характеристический определитель.